Using the Price-to-Earnings Harmonic Mean to Improve Firm Valuation Estimates

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This paper reviews some well-known options to estimate the portfolio average of the price-earnings (P/E) multiple, emphasizing the logic and calculation of the harmonic mean. The harmonic mean is a useful but oftentimes unfamiliar calculation to many students and professionals. The simple arithmetic mean when applied to non-price normalized ratios such as the P/E is biased upwards and cannot be numerically justified, since it is based on equalized earnings. The paper advocates the use of the classic harmonic mean when the need is for an equal-dollar-weighted average and the weighted-average harmonic mean when the need is for an index style market-weighted average.

INTRODUCTION

A frequently used method by financial analysts to value a firm is the comparables approach, a technique resting on the logic of the law of one price–substitutes should sell at a similar relative price. In this approach, market multiples such as market-to-book, EBITDA to value, price to cash flow, price to sales and, most commonly, price-earnings (P/E) are often used. Subsequently, these individual multiples are aggregated at the portfolio level to provide a unified number. Herein lies the problem–is there a best way or even an accepted way to average these individual multiples?

When asked to compute an average, many students as well as practitioners assume the arithmetic mean is what is called for; often they are not aware that there are better alternative approaches to capture the central tendency of a sample. In the case of relative valuation (i.e., the comparables approach) the choice of averaging method does matter. As we will show later, using the arithmetic mean creates a consistent upward bias in valuations. On the other hand, the much less familiar harmonic mean provides a more logical approach to averaging valuation multiples such at P/E. Students, instructors, and practitioners interested in valuation need to be familiar with this important tool. The harmonic mean is not a new calculation by any means and it is used by some practitioners, but it is not discussed in many finance/valuation textbooks and for some it may be a forgotten or unfamiliar calculation.

The purpose of this paper is to introduce students, instructors, and practitioners to the calculation and logic of the harmonic mean and specifically to illustrate its value when averaging industry P/E ratios for a firm valuation. First we review what averaging methods have been used and suggested in a sampling of textbooks, in academic research, and in the industry. Then we introduce the harmonic mean and illustrate its relevance when averaging ratios such as the P/E ratio used in firm valuations. We also discuss variants of the harmonic mean as used in the industry for calculating average index P/Es.

METHODS OF AVERAGING USED IN PRACTICE

For the serious student and the practitioner of finance, it is not such a simple matter to take an average because one must choose a methodology to calculate an average (see Table 1 for various methods used to calculate an average P/E). The confusion this creates is certainly not new. Back in 1886 Coggeshall addressed this issue in the *Quarterly Journal of Economics*: "The mean commonly employed by the economist is . . . not a real quantity at all, but it is a quantity assumed as the representative of a number of others differing from it more or less . . . . This is the fictitious mean or average, properly so called. Its fictitious character renders it possible to make choice among different values, and thus among different methods of finding it" (Coggeshall, 1886, p.84).

The academic work that uses ratios for valuation, our textbooks, and even the industry offer mixed guidance, or often no guidance on the appropriate method to use. Consider several well-known academic studies which calculate average P/E ratios. Kim and Ritter (1999), in a study of IPO pricing, use the median and the geometric mean of comparable firm P/Es. In other studies, the median is sometimes used (e.g., Cheng and McNamara 2000, Alford 1992), as is the simple arithmetic mean (e.g., Lie and Lie 2002, Beatty, Riffe, Thompson 1999), and the harmonic mean (e.g., Liu, Nissim, Thomas 2002, Beatty, Riffe, Thompson 1999).

The textbooks and practitioner's handbooks offer no greater consistency or guidance. Benjamin Graham, an early proponent of using P/E ratios for analyzing stock prices, uses the arithmetic mean in *The Interpretation of Financial Statements* (1937) but the harmonic mean in *Security Analysis* (1951). Most simply advise as Link and Boger (1999) do: "an average ratio can be calculated" with no further elucidation (p. 82). Titman and Martin in their *Valuation* text (2008) never specify the method of obtaining the average multiple (they use P/E and EBITDA multiples), but their examples use the simple arithmetic average (see p. 230 and 237). Berk and Demarzo do not specify how they obtained or calculated the industry average multiples used in their example in their *Corporate Finance* text (2007, p. 625). In the example in their *Fundamentals of Corporate Finance* text (2009) they use simple arithmetic averages (p. 307).

In *Applied Corporate Finance* (2006), Damodaran does not define the average used, but his example (p. 584) uses a simple arithmetic average. In his valuation text *Damodaran on Valuation (2006, pp. 241-244)*, Damodaran has a lengthy discussion of which average to use. He advocates using the median instead of the mean for an industry average to offset the positive skewness he finds in P/E distributions (i.e., the mean will

Table 1. Equations for Calculating the Average P/E Ratio

Averaging Method	Equation
Arithmetic($\overline{PE}_A$)	$\overline{PE}_A = \frac{1}{N}\sum_{i=1}^{N}\frac{P_i}{EPS_i}$
Geometric($\overline{PE}_G$)	$\overline{PE}_G = \left[\prod_{i-1}^{N}\frac{P_i}{EPS_i}\right]^{\frac{1}{2}}$ or $\overline{PE}_G = EXP\left[\frac{1}{N}\sum_{i=1}^{N}\ln\left(\frac{P_i}{EPS_i}\right)\right]$
Harmonic or equal dollar weighted ($\overline{PE}_{HM}$)	$\overline{PE}_{HM} = \frac{1}{\frac{1}{n}\sum_{i=1}^{N}\frac{EPS_i}{P_i}}$ *or* $\overline{PE}_{HM} = \frac{n}{\frac{1}{n}\sum_{i=1}^{N}\frac{EPS_i}{P_i}}$
Price - weighted ($\overline{PE}_{PW}$)	$\overline{PE}_{PW} = \frac{\sum_{i-1}^{N} P_i}{\sum_{i=1}^{N} EPS_i}$
Weighted - average ($\overline{PE}_{WT}$)	$\overline{PE}_{WT} = \sum_{i-1}^{N}\left[w_i \bullet \frac{P_i}{EPS_i}\right]$

be greater than the median). Further, he suggests using "the inverse of the price-earnings ratio . . . the earnings yield" to handle the problem of firms with negative earnings which do not have a meaningful P/E. As we will shortly demonstrate, this is the harmonic mean, but he suggests it for a different reason (the problem of negative P/Es) than what is addressed here. Welch (2009, p. 514) has a somewhat similar discussion to

Damodaran's. Welch suggests solving the negative P/E issue by using the median or averaging the earnings yield and inverting (e.g., harmonic mean). Stowe, Robinson, Pinto, and McLeavey (2002), in the required valuation text for the chartered financial analyst exam, also say the same thing. Joyce and Roosma (1991) advocate using median ratios "to avoid the distorting effect of extremes on the arithmetic average" (p. 32.5). Thus some texts go beyond simply using the arithmetic mean to include in their discussions medians and a correction for negative P/Es, but none discuss the inherent drawbacks to the arithmetic mean.

And what is used in the industry? The Investment Company Institute, the mutual funds trade group, says funds may use any method they like to calculate P/E ratios. Spokesman Chris Wloszczyna says, "There are no regulations." At Vanguard, the industry's second-largest fund company, spokesman Brian Mattes says he's "not sure what methodology" the company uses to determine its funds' P/E ratios.[1] Value Line reports an average P/E that is the arithmetic mean, but with negative P/Es and those greater than 100 excluded.[2]

Clearly there does not appear to be any standard or consistency among either academics or practitioners. However, there has been research attempting to determine empirically the best measure. Baker and Rubak (1999) suggest that the harmonic mean should be used when estimating a single industry multiple. Liu, Nissam, and Thomas (2002) find that using the harmonic mean improves performance relative to the simple mean or median.

What properties make the harmonic mean an arguably better method of averaging for certain financial ratios? The simple and familiar example which follows motivates the discussion.

THE CLASSIC SPEED EXAMPLE—WHEN THE HARMONIC MEAN MAKES SENSE

Here is the classic "average speed" problem. If your vehicle averaged 60 mph in the first hour, how much distance will you cover in 2 hours? The answer is 120 miles, assuming that you maintain the same average during the second hour. Continuing with the problem, if your average speed from New York to Boston was 60 mph and the average speed on the return trip was 20 mph, what was your average speed roundtrip? The answer is *not* 40 mph, because the travel time over the same distance varies over the two legs. This is a common problem associated with averaging ratios that have two independent variables in the numerator and the denominator (distance and time in this case).

Distance (D) traveled equals speed (S) multiplied by time (T). In this problem we have $S_1 = 60$ mph, $S_2 = 20$ mph, and $D_1 = D_2 = D$. Time on each leg of the trip is therefore $T_1 = D/60$ and $T_2 = D/20$. The average roundtrip speed S_A equals the total distance (2D) traveled divided by the total elapsed time:

$$S_A = \frac{2D}{\dfrac{D}{60\text{ mph}} + \dfrac{D}{20\text{ mph}}}$$

Rearranging,

$$S_A = \frac{1}{\left[\dfrac{1}{2} \times \left(\dfrac{D}{60\text{ mph}} + \dfrac{D}{20\text{ mph}}\right)\right]} = 30\text{ mph}$$

What we just applied here is the concept of the harmonic mean to average speeds over the same distance traveled. We separated the two independent variables, distance and time, added them up separately, and then formed a ratio--notice that the denominator of Equation 1 is in time units only. The harmonic mean provides the correct answer.

COMPARING THE ARITHMETIC MEAN AND THE HARMONIC MEAN OF P/E RATIOS

In the example above we see how a simple arithmetic mean can produce the wrong answer when dealing with driving time and distance. But does that mean the arithmetic mean of P/E ratios also gives a deceptive result? Consider this example. Say you are averaging the P/E of two stocks, one with EPS of $10 (Stock A) and one with EPS of $20 (stock B). Imagine Stock A has a price of $100, resulting in a P/E of 10. Stock B has a price of $60, resulting in a P/E of 3. The straight arithmetic average of the two P/Es is obviously 6.5. But notice that the way to obtain this arithmetic mean P/E value is to buy two shares of A for every share of B, to equalize the earnings accruing from each holding. If you buy two shares of A for a $200 investment resulting in $20 of earnings, and one share of B for a $60 investment and $20 of earnings, the portfolio P/E becomes $260/$40 = 6.5. However, if the portfolio has only one share of each A and B, the total value of the investment is $100 + $60 resulting in total earnings of $10 + $20. Then the realized portfolio P/E is in fact $160/$30 = 5.33, the price-weighted average P/E. (See Table 2 for all results for this simple portfolio.)

It is important to realize that the simple arithmetic average of a P/E ratio does not give equal weight to each share (e.g., one share per firm), nor does it assume an equal dollar investment in each firm; instead, it gives equal weight to the *earnings* in each ratio. Thus, in terms of P/E the arithmetic mean automatically assumes that you are investing more dollars in the stock with the higher P/E. In our example, stock A has a P/E of 10 and stock B has a P/E of 3; you invest 200/60 or 3.33 times as many dollars in stock A as stock B, to equalize the $20 of earnings accruing from each position. (See Appendix A for a proof of this relationship with a two stock portfolio.) While the simplicity of the

Table 2. Two Stock Portfolio Mean Calculation Results

	Stock A	Stock B
Price	100	60
Earnings	10	20
EPS	10	20
P/E	10	3
Arithmetic Mean	**6.5**	
Harmonic Mean	**4.62**	
Weight: One share of each		
Number of shares	1	1
Investment	100	60
Earnings	10	20
Portfolio P/E (100+60)/(10+20)	**5.33**	
Weight: Equal earnings		
Number of shares	2	1
Investment	200	60
Earnings	20	20
Portfolio P/E (200+60)/(20+20)	**6.5**	
Weight: Equal investment in each firm		
Number of shares	6	10
Investment	600	600
Earnings	60	200
Portfolio P/E (600+600)/(60+200)	**4.62**	

Note: It is clear that the arithmetic mean P/E equals the P/E of an equal earnings portfolio, and the harmonic mean P/E equals the mean of an equal investment portfolio. A portfolio of one share each produces yet a third P/E, the price-weighted average.

arithmetic mean may appear to be convenient, the convenience comes at the price of inaccuracy and can be considered misleading.

Rather than average price per unit of earnings (the simple arithmetic average), the harmonic mean averages the inverse, called the earnings yield (earnings per unit of price). The importance and intuition of this can be expressed as follows: We use ratios in financial analysis to control for size. That is, we cannot directly compare net income of firm A with net income of firm B if the firms are of different size. So we compare instead ROA (net income scaled by assets) or ROE (net income scaled by equity). Still considering the PE ratio:

$$\frac{P}{E} = \frac{\text{Price Per Share}}{\text{Earnings Per Share}} = \frac{\text{MVE}}{\text{NI}}$$

where MVE is market value of equity and NI is net income. This is obviously the inverse of a size-scaled variable (net income scaled by total equity). The harmonic mean computes the average of this size-scaled variable (E/P or, below, EPS_i/P_i)).

$$\overline{PE}_{HM} = \frac{1}{\frac{1}{n}\sum_{i-1}^{n}\frac{EPS_i}{P_i}} \quad \text{or, equivalently,} \quad \overline{PE}_{HM} = \frac{n}{\frac{1}{n}\sum_{i-1}^{n}\frac{EPS_i}{P_i}}$$

By averaging the yields, the harmonic mean is equivalent to a portfolio P/E assuming an equal amount invested in each firm, a much more straightforward interpretation than the one for the arithmetic mean. To see this, let's return to our example: stock A (P = 100, EPS = 10, P/E = 10) and stock B (P = 60, EPS = 20, P/E = 3). The harmonic mean is

$$\frac{1}{\frac{1}{2}\left(\frac{1}{3}+\frac{1}{10}\right)} = 4.62$$

To demonstrate that the harmonic mean is equivalent to investing an equal dollar amount in each stock, we might invest $600 in each stock (six shares of stock A and 10 shares of stock B). The P/E of this portfolio would be (600 + 600)/[(6 x 10) + (10 x 20)] = 4.62.

In our example the arithmetic average P/E (an equal earnings portfolio) is 6.5, the price-weighted average P/E (a portfolio of one share of each firm) is 5.33, and the harmonic mean (equivalent to equal investment in each firm) is 4.62. Table 3 contains all the mean results for our two stock portfolio based on the formulas in Table 1. The results are ranked from lowest to highest average P/E: the arithmetic mean > the geometric mean > the harmonic mean. In Appendix B we show that the arithmetic mean is unequivocally biased upwards versus the harmonic mean (the appendix also includes the geometric mean).[3] In this case, the price-weighted mean (one share of each stock) and the weighted-average mean come in highest, but that will not always be the case. Thus the harmonic mean is valuable because it solves the upward bias problem encountered while using arithmetic means and also has a much more intuitive and logical

Table 3. Estimates of Average for Sample Two-Stock Portfolio

Estimates of mean for a two-stock portfolio consisting of a \$100 stock with \$10 EPS and a \$60 stock with a \$20 EPS.

Method	Estimate of Mean
Harmonic or equal dollar weighted	4.62
Price-weighted or equal number of shares	5.33
Geometric	5.48
Arithmetic or equal earnings weighted	6.5
Weighted-average	7.38

investment assumption than the arithmetic mean, since most portfolios are neither on an equal number of shares per position basis, nor on an equalized earnings per holding basis.

THE HARMONIC MEAN P/E RATIO OF STOCK INDICES

The method illustrated so far for calculating the harmonic mean P/E ratio assumes the analyst desires an average P/E for any group of stocks, such as an industry average to use for valuation of a target firm. The harmonic mean assumes an *equal dollar weighting*. It is possible to adapt the harmonic mean for other weightings such as those found in stock indices. Calculating the aggregate P/E of a capitalization-weighted index requires a minor modification to the standard harmonic mean formulation. The solution used in practice is to use the weighted harmonic mean.

$$\overline{\mathrm{PE}}_{\mathrm{WtHM}} = \frac{1}{\sum_{i-1}^{N}\left[w_i \bullet \frac{EPS_i}{P_i}\right]}$$

where w_i is the weight of the stock in the index (and $\sum w_i = 1$). If the weights do not total 1 (which they might not if the weights were calculated on the basis of the entire market rather than a smaller stock index under examination), then:

$$\mathrm{PE}_{\mathrm{WtHM}} = \frac{\sum_{i=1}^{N} w_i}{\sum_{i-1}^{N}\left[w_i \bullet \frac{EPS_i}{P_i}\right]}$$

Because the Dow Jones Industrial Average (DOW) is calculated assuming an investment of one share in each stock, it would seem likely that the index P/E would be calculated on a price-weighted basis, resulting in unequal dollar amounts invested per stock. However, that is not the assumption Dow Jones uses to calculate the average P/E for the DJIA. The formula employed by Dow Jones is:[4]

$$\text{DJIA Average PE} = \frac{\sum_{i-1}^{N}[\text{NUM}_i \bullet \text{P}_i]}{\sum_{i=1}^{N}[\text{NUM}_i \bullet \text{EPS}_i]}$$

where NUM is the number of (adjusted) shares outstanding. This is a value-weighted average and is equivalent to the weighted average harmonic mean in equation 4.[5]

The various index averaging methods will become clearer if we continue our example from Section IV. Again imagine a two-stock portfolio consisting of shares of stock A, a \$100 stock with \$10 EPS, and shares of stock B, a \$60 stock with a \$20 EPS. For a constant portfolio value of \$1,000 the various central measures of P/E are presented in Table 4. Note that the standard harmonic mean produces the lowest P/E estimate, while the arithmetic mean produces the highest.

Recall that the average P/E is oftentimes used by analysts to value a firm or its share price. Share price equals the comparable P/E times the firm's earnings per share (or, equivalently, firm equity value equals the comparable P/E times the firm's net income). If the analyst were using these estimates to value a stock, and the firm under consideration had earnings per share of \$15, the results would range from \$69.3 (harmonic mean of 4.62 x \$15) to \$97.5 (arithmetic mean of 6.5 x \$15), a substantial difference of 40%. Arguably, the harmonic mean provides the most conservative and applicable P/E estimate.

CONCLUSION

Although the harmonic mean is not a particularly popular method to measure central tendency, this paper suggests that it is the more appropriate measure. This is particularly so when obtaining meaningful and applicable industry P/E averages to back out individual firm valuations. The harmonic mean is appropriate when averaging ratios that have independent variables in the numerator and the denominator, like the P/E ratio does; the harmonic mean is appropriate when there is the possibility of nonsensical or meaningless ratios such as a negative P/E; and the harmonic mean is not biased upwards like the arithmetic mean is. The harmonic mean implicitly makes the assumption of an equal dollar investment in each stock, a more reasonable assumption than equal earnings from each stock, the assumption of the arithmetic mean. The formula can be easily

Table 4. P/E Means for $1,000 Portfolio

	Allocations		Shares		Portfolio		
Portfolio Type	A	B	A	B	Value ($)	P/E	Averaging Method
Equal Weight	50.00%	50.00%	5.00	8.33	1,000.00	4.62	Harmonic Mean
Market Cap Weight	67.57%	32.43%	6.76	5.41	1,000.00	5.69	Weighted-Average Harmonic Mean
Price Weighted (Equal Shares)	62.50%	37.50%	6.25	6.25	1,000.00	5.33	Price-Weighted Mean
Equal Earnings	76.90%	23.08%	7.69	3.846	1,000.00	6.50	Arithmetic Mean

Stock Price ($)	100	60
EPS	10	20
P/E	10	3
Shares Outstanding	50	40
Market Cap ($)	5000	2400

applied to other ratios used in valuations such as market to book, price to cash flow, or price to sales. At the index level, simple variations of the harmonic mean are used when the need is for an average P/E measure of the index.

ENDNOTES

[1] Both quotations are from Berenson 1997.

[2] Value Line provided this information in response to an information request. The *Value Line Investment Analyzer* does not explain how it calculates portfolio P/Es.

[3] Estimators like the median or the value-weighted mean can lie anywhere above, below, or within the range of the harmonic and arithmetic means (Baker and Rubak, 1999).

[4] This calculation cannot be found on the Dow Jones website but was conveyed in response to an inquiry. Dow Jones uses EPS figures from Factset Research Systems and would not share that data. They use the latest 12 months of reported EPS available.

[5] To see that equation 5 equals equation 4, let the total market value of the index (or portfolio) equal TOT. Divide the numerator and denominator of equation 5 by TOT:

$$\frac{\sum_{i=1}^{N}\left[\frac{NUM_i}{TOT}\bullet P_i\right]}{\sum_{i=1}^{N}\left[\frac{NUM_i}{TOT}\bullet EPS_i\right]}=\frac{\sum_{i=1}^{N} w_i}{\sum_{i=1}^{N}\left[\frac{NUM_i\bullet P_i}{TOT}\bullet\frac{EPS_i}{P_i}\right]}=\frac{\sum_{i=1}^{N} w_i}{\sum_{i=1}^{N} w_i\bullet\frac{EPS_i}{P_i}}$$

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Appendix A

Proof: Arithmetic mean assumes an investment proportional to the P/E ratios.

Assume a two stock portfolio. Let y = the multiple of the amount invested in stock A that you invest in stock B. Thus the number of shares purchased of B, if you buy one share of A, is $\frac{y \cdot P_A}{P_B}$.

Set arithmetic mean P/E equal to the P/E of this portfolio.

$$\frac{\frac{P_A}{E_A} + \frac{P_B}{E_B}}{2} = \frac{P_A + \frac{y \cdot P_A}{P_B} \cdot P_B}{E_A + \frac{y \cdot P_A}{P_B} \cdot E_B}$$

Rearrange:

$$\frac{P_A}{E_A}\left(E_A + \frac{y \cdot P_A \cdot E_B}{P_B}\right) + \frac{P_B}{E_B}\left(E_A + \frac{y \cdot P_A \cdot E_B}{P_B}\right) = 2 \cdot P_A + 2 \cdot y \cdot P_A$$

Solve for y:

$$y = \frac{E_A \cdot P_B}{E_B \cdot P_A} = \frac{P_B/E_B}{P_A/E_A} = \frac{P/E_B}{P/E_A}$$

Thus the amount invested in stock B is a multiple of the amount invested in stock A, and is determined by the ratio of the two P/E ratios.

Appendix B

Proof: arithmetic mean (AM) ≥ geometric mean (GM) ≥ harmonic mean (HM). For all X_i greater than zero.

Note:

$$AM = \frac{1}{n}\sum X_i \quad and \ \ln AM = \ln\left(\frac{1}{n}\sum X_i\right)$$

$$GM = \left[\prod X_i\right]^{1/n} \quad and \ \ln GM = \frac{1}{n}\sum \ln X_i$$

$$HM = \frac{1}{\frac{1}{n}\sum \frac{1}{X_i}} = \left[\frac{1}{n}\sum \frac{1}{X_i}\right]^{-1} \quad and \ \ln HM = -\ln\left(\frac{1}{n}\sum \frac{1}{X_i}\right)$$

By Jensen's inequality,

$$\ln AM = \ln\left(\frac{1}{n}\sum X_i\right) \geq \frac{1}{n}\sum \ln X_i = \ln GM$$

Thus AM ≥ GM.

Similarly, also by Jensen's inequality,

$$\ln HM = -\ln\left(\frac{1}{n}\sum \frac{1}{X_i}\right) \leq -\frac{1}{n}\sum \ln \frac{1}{X_i} = \frac{1}{n}\sum \ln X_i = \ln GM$$

$Thus\, GM \geq HM\, and\, AM \geq GM \geq HM.$